SOLUTION OF INHOMOGENEOUS HEAT-TRANSFER LINES USING THE METHOD OF THE THEORY OF CHAINS

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UDC 536.21

A derivation is given and finite expressions are presented for the A-parameters for inhomogeneous heat lines; examples are given of a computation using the formulas obtained.

Let a heat line be a heat object in which the change in temperature is determined completely by its change along one radius in the direction of heat propagation. Such heat objects as a wall, a cylinder, a sphere can be considered as heat lines. An inhomogeneous heat line is considered to be one for which at least one of the parameters (heat conductivity, volume specific heat, or cross-sectional area) is a function of the line coordinates. In the most general case all three parameters can vary simultaneously and independently along the line. The cylinder and sphere are numbered among inhomogeneous systems because such a parameter as the cross section varies along the radius in such bodies.

The purpose of the present paper is to show that it is possible to obtain solutions in general form (independently for the temperature and the heat flux) for an inhomogeneous heat line with any given boundary conditions written in the so-called A-parameters. Insofar as we know, A-parameters are presently obtained and used only for homogeneous heat lines in the heat-engineering literature [1].

Indubitably, the expressions obtained for A-parameters for inhomogeneous heat lines can also be used to compute inhomogeneous acoustic lines as well as other unidirectional transfer processes such as mass transfer due to concentration gradient, etc.

It is convenient to use the methods of chain theory in seeking the solutions of one-dimensional problems of heat kinetics. With their aid it is extremely simple to find the steady reaction on an applied harmonic effect. Quite special parameters, called waves, are used in this theory: $\gamma = \alpha + i\beta$ is the propagation constant (where α is the damping coefficient, and β the phase coefficient), and ρ is the wave resistance expressed in terms of the linear line parameters $c_l = c_V(x)S(x)$ and $r_l = 1/\lambda(x)S(x)$ as follows:

$$\gamma(x) = \sqrt{i\omega c_l r_l}; \quad \rho(x) = \sqrt{\frac{r_l}{i\omega c_l}}.$$

It is clear that for inhomogeneous heat lines both the propagation constant and wave resistance vary continuously along the coordinate x.

Chain theory is used together with the methods of solving differential equations but it also has advantages of its own. Chain theory methods are simpler in mathematical respects, and much more graphic when considering complex systems. Chain theory is used to synthesize new systems corresponding to given requirements (specified energy attenuation, specified lag time) and to analyze the solutions obtained.

The method we selected for seeking the A-parameters of an inhomogeneous heat line is to partition the selected line into n sections, to write the A-parameters for each section by considering the parameters of each section to be concentrated, and to find the exact values of the A-parameters of the whole line, which are obtained in the limit when the number of sections becomes infinite.

Institute of Heat and Mass Transfer, Academy of Sciences of the Belorussian SSR, Minsk. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 19, No. 5, pp. 936-940, November, 1970. Original article submitted January 8, 1970.

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Let us find the A-parameters of an inhomogeneous heat line considered as a quadripole, when the linear parameters c_l and r_l are arbitrary, but vary continuously along the line.

After the partition, if the total length of the line is l, the length of one section is written as

$$\Delta x = \frac{l}{n}.$$

In the case of a sufficiently small length Δx, we have for the section k

$$\begin{bmatrix} A_k & B_k \\ C_k & D_k \end{bmatrix} \cong \begin{bmatrix} 1 & r_{lk}\Delta x \\ sc_{lk}\Delta x & 1 \end{bmatrix} \cong J + \Delta x X_k, \tag{1}$$

$r_{lk} = r_l(k\Delta x)$; $c_{lk} c_l(k\Delta x)$; $k = j, 2, \ldots, n$,

$$J = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; \quad X_k = \begin{bmatrix} 0 & r_{lk} \\ sc_{lk} & 0 \end{bmatrix}.$$

The A-parameters of the whole line $n \to \infty$ ($\Delta x \to 0$);

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \lim_{n\to\infty} \prod_{k=1}^{n} \left(J + \frac{l}{n} X_k \right) = \lim_{n\to\infty} \left[J + \frac{l}{n} \sum_{k=1}^{n} X_k \right.$$
$$+ \left(\frac{l}{n}\right)^2 \sum_{i<j} \sum X_i X_j + \left(\frac{l}{n}\right)^m \sum_{i_1 < i_2} \sum \ldots \sum_{\ldots < i_m} X_{i_1} X_{i_2} \ldots X_{i_m}$$
$$\left. \ldots + \left(\frac{l}{n}\right)^n \sum_{i_1 < i_2} \sum \ldots \sum_{\ldots < i_n} X_{i_1} X_{i_2} \ldots X_{i_n} \right]. \tag{2}$$

Let us replace the sum by an integral

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = J + \int_0^l X(x)\,dx + \int_0^l \int_0^{x_2} X(x_1)\, X(x_2)\, dx_1 dx_2 + \ldots$$
$$\ldots + \int_0^l \int_0^{x_n} \int_0^{x_{n-1}} \cdots \int_0^{x_2} X(x_1) X(x_2) \ldots X(x_n)\, dx_1 dx_2 \ldots dx_n. \tag{3}$$

But

$$X(x_1) X(x_2) \ldots X(x_{2n}) = s^n \begin{bmatrix} r_l(x_1) c_l(x_2) \ldots r_l(x_{2n-1}) c_l(x_{2n}) & 0 \\ 0 & c_l(x_1) r_l(x_2) \ldots c_l(x_{2n-1}) r_l(x_{2n}) \end{bmatrix}, \tag{4}$$
$$n = 1, 2, \ldots,$$

$$X(x_1) X(x_2) \ldots X(x_{2n+1}) = s^n \begin{bmatrix} 0 & r_l(x_1) c_l(x_2) \ldots c_l(x_{2n}) r_l(x_{2n+1}) \\ sc_l(x_1) r_l(x_2) \ldots r_l(x_{2n}) c_l(x_{2n+1}) & 0 \end{bmatrix}, \tag{5}$$
$$n = 0, 1, 2, \ldots$$

and the exact expressions for the A-parameters are

$$A = 1 + \sum_{n=1}^{\infty} a_n s^n, \tag{6}$$

$$B = \int_0^l r_l(x)\,dx + \sum_{n=1}^{\infty} b_n s^n, \tag{7}$$

$$C = s \int_0^l c_l(x)\,dx + s \sum_{n=1}^{\infty} c_n s^n, \tag{8}$$

$$D = 1 + \sum_{n=1}^{\infty} d_n s^n, \tag{9}$$

where

$$a_n = \int_0^l c_l(x_{2n}) \int_0^{x_{2n}} r_l(x_{2n-1}) \int_0^{x_{2n-1}} c_l(x_{2n-2}) \dots \int_0^{x_3} c_l(x_2) \int_0^{x_2} r_l(x_1) dx_1\, dx_2 \dots dx_{2n}, \tag{10}$$

$$b_n = \int_0^l r_l(x_{2n+1}) \int_0^{x_{2n+1}} c_l(x_{2n}) \int_0^{x_{2n}} r_l(x_{2n-1}) \dots \int_0^{x_3} c_l(x_2) \int_0^{x_2} r_l(x_1) dx_1\, dx_2 \dots dx_{2n+1}, \tag{11}$$

$$c_n = \int_0^l c_l(x_{2n+1}) \int_0^{x_{2n+1}} r_l(x_{2n}) \int_0^{x_{2n}} c_l(x_{2n-1}) \dots \int_0^{x_3} r_l(x_2) \int_0^{x_2} c_l(x_1) dx_1\, dx_2 \dots dx_{2n+1}, \tag{12}$$

$$d_n = \int_0^l r_l(x_{2n}) \int_0^{x_{2n}} c_l(x_{2n-1}) \int_0^{x_{2n-1}} r_l(x_{2n-2}) \dots \int_0^{x_3} r_l(x_2) \int_0^{x_2} c_l(x_1) dx_1\, dx_2 \dots dx_{2n}. \tag{13}$$

It is seen from the expressions presented that the A-parameters are represented as power series in the Laplace operator the coefficients of which are given as integrals of a function of the linear heat resistance and the linear specific heat. Any system function of an inhomogeneous heat line, closed by means of an arbitrary resistance, can be determined by using A-parameters.

The correctness of the expressions representing the A-parameters is confirmed by the solutions we obtained on their basis for exponential and linear heat lines heat-insulated from the end (i.e., lines for which both the linear heat resistance and the linear specific heat vary along the line by an exponential and a linear law, respectively). The solutions obtained agreed with solutions existing in the literature.

Expressions for the A-parameters for inhomogeneous heat lines written in terms of the generalized wave parameters of a line, mentioned at the beginning, could also be obtained.

In order to demonstrate the ultilization of the formulas obtained, let us determine the value of the parameter A for a wall, a cylinder, and a sphere.

For the case of a homogeneous line (wall), the value of the A-parameter is obtained at once (the length of the line is assumed equal to one):

$$A = \sum_{n=0}^{\infty} \frac{(sc_{l0} r_{l0})^n}{2n!} = ch \sqrt{sc_{l0} r_{l0}} = chz, \tag{14}$$

where $z = \sqrt{sc_{v0}/\lambda_0}$.

In the case of the assignment of a thermal perturbation on the surface of a cylinder (or sphere), knowledge of the A-parameter is completely sufficient for the determination of the temperature at any point along the radius of the body under consideration.

Let us evaluate the A-parameter for a cylinder the maximum radius of which is taken equal to one:

$$A = 1 + a_1 s + a_2 s^2 + \dots,$$

$$a_1 = \int_0^1 c_l(x_2) \int_0^{x_2} r_l(x_1)\, dx_1 dx_2 .$$

Since the beginning of the heat line is the cylinder surface, then the coordinate considered along the line will pass through the following values: $1, x_1, x_2, \dots, 0$. Then

$$a_1 = \int_1^0 c_{v0} 2\pi x_2 \int_1^{x_2} \frac{1}{2\pi\lambda_0 x_1}\, dx_1 dx_2 = \frac{1}{4}\frac{c_{v0}}{\lambda_0}, \quad a_1 s = \frac{(z/2)^2}{(1!)^2}.$$

Proceeding in the same manner, we obtain

$$a_2 s^2 = \frac{(z/2)^4}{(2!)^2} \text{ and } a_3 s^3 = \frac{(z/2)^6}{(3!)^2} \text{ etc.} \tag{15}$$

In other words, $A = I_0(z)$, where $I_0(z)$ is the zero-order Bessel function of argument z.

Now, let us determine the parameter A for a sphere

$$a_1 = \int_l^0 c_{v0} 4\pi x_2^2 \int_l^{x_2} \frac{1}{4\pi\lambda_0 x_1^2} dx_1 dx_2 ,$$

$$a_1 = \frac{1}{6}\frac{c_{v0}}{\lambda_0}; \qquad a_1 s = \frac{z^2}{3!} .$$

Proceeding in the same manner we obtain

$$a_2 s^2 = \frac{z^4}{5!}; \quad a_n s^n = \frac{z^{2n}}{(2n+1)!} \text{ and } A = 1 + \sum \frac{z^{2n}}{(2n+1)!} = \frac{shz}{z} . \tag{16}$$

It must be noted that the exact solution of the heat conduction differential equation is not sought for any inhomogeneous heat lines but only for those for which the linear heat resistance and linear volume specific heat functions are interrelated in a definite manner. These constraints are absent in the method proposed.

It is evident that knowledge of the A-parameters of inhomogeneous heat lines permits significant expansion of the circle of problems solved by the use of quadrupoles, which is limited at this time to the analysis of homogeneous heat lines [1].

NOTATION

x	is the coordinate, m;
S	is the cross-sectional area, m^2;
s	is the Laplace operator;
ω	is the angular frequency, rad/sec;
λ	is the coefficient of heat conduction, W/m · °C;
c_V	is the volume specific heat, $J/m^3 \cdot {}^\circ C$;
c_l	is the linear specific heat, J/m · °C;
r_l	is the linear heat resistance, °C/W · m.

LITERATURE CITED

1. R. Maillard, Revue Generale de Thermique, 8, 90 (1969).